Nonlocal Hamiltonian operators of hydrodynamic type with flat metrics, integrable hierarchies and the equations of associativity[1]

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1 Introduction

In this paper we solve the problem of describing all nonlocal Hamiltonian operators of hydrodynamic type with flat metrics and establish that this nontrivial special class of Hamiltonian operators is closely connected with the associativity equations of two-dimensional topological quantum field theories and the theory of Frobenius manifolds. It is shown that the Hamiltonian operators of this class are of special interest for many other reasons too. In particular, we prove in this paper that any such Hamiltonian operator always defines integrable structural flows (systems of hydrodynamic type), always gives a nontrivial pencil of compatible Hamiltonian operators and generates integrable hierarchies of hydrodynamic type. It is proved that the affinors of any such Hamiltonian operator generate some special integrals in involution. The nonlinear systems describing integrals in involution are of independent great interest. The equations of associativity of two-dimensional topological quantum field theories (the Witten–Dijkgraaf–Verlinde–Verlinde and Dubrovin equations) describe an important special class of integrals in involution and a special class of nonlocal Hamiltonian operators of hydrodynamic type with flat metrics. It is shown that any N-dimensional Frobenius manifold can be locally presented by a certain special flat N-dimensional submanifold with flat normal bundle in a $2N$-dimensional pseudo-Euclidean space and this submanifold is defined uniquely up to motions. We will devote a separate paper to the properties of this construction and to the properties of this special class of flat submanifolds with flat normal bundle (we mean the class corresponding to Frobenius manifolds).

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We recall that the general nonlocal Hamiltonian operators of hydrodynamic type, namely, the Hamiltonian operators of the form

$$P^{ij} = g^{ij}(u(x))\frac{d}{dx}+b^{ij}_k(u(x))\, u^k_x+\sum_{n=1}^{L} \varepsilon^n (w_n)^i_k(u(x))u^k_x \left(\frac{d}{dx}\right)^{-1} \circ (w_n)^j_s(u(x))u^s_x, \quad (1.1)$$

where $\det(g^{ij}(u)) \neq 0$, $\varepsilon^n = \pm 1$, $1 \leq n \leq L$, $u^1, \ldots, u^N$ are local coordinates, $u = (u^1, \ldots, u^N)$, $u^i(x)$, $1 \leq i \leq N$, are functions (fields) of one independent variable x, the coefficients $g^{ij}(u)$, $b^{ij}_k(u)$, $(w_n)^i_j(u)$, $1 \leq i, j, k \leq N$, $1 \leq n \leq L$, are smooth functions of local coordinates, were studied by Ferapontov in the paper [1] in connection with vital necessities of the Hamiltonian theory of systems of hydrodynamic type (see also [2], [3]).

The Hamiltonian operators of the general form (1.1) (local and nonlocal) play a key role in the Hamiltonian theory of systems of hydrodynamic type. We recall that an operator P^{ij} is said to be *Hamiltonian* if the operator defines a Poisson bracket

$$\{I, J\} = \int \frac{\delta I}{\delta u^i(x)} P^{ij} \frac{\delta J}{\delta u^j(x)} dx \quad (1.2)$$

for arbitrary functionals I and J on the space of the fields $u^i(x)$, i.e. the bracket (1.2) is skew-symmetric and satisfies the Jacobi identity.

It is proved in the paper [1] that the operator (1.1) is Hamiltonian if and only if $g^{ij}(u)$ is a symmetric (pseudo-Riemannian) contravariant metric and also the coefficients of the operator satisfy the following relations:

1) $b^{ij}_k(u) = -g^{is}(u)\Gamma^j_{sk}(u)$, where $\Gamma^j_{sk}(u)$ is the Riemannian connection generated by the contravariant metric $g^{ij}(u)$ (the Levi-Civita connection),

2) $g^{ik}(u)(w_n)^j_k(u) = g^{jk}(u)(w_n)^i_k(u)$,

3) $\nabla_k (w_n)^i_j(u) = \nabla_j (w_n)^i_k(u)$, where ∇_k is the operator of covariant differentiation generated by the Levi-Civita connection $\Gamma^j_{sk}(u)$ of the metric $g^{ij}(u)$,

4) $R^{ij}_{kl}(u) = \sum_{n=1}^L \varepsilon^n \left((w_n)^i_l(u)(w_n)^j_k(u) - (w_n)^j_l(u)(w_n)^i_k(u)\right)$, where

$$R^{ij}_{kl}(u) = g^{is}(u)R^j_{skl}(u)$$

is the Riemannian curvature tensor of the metric $g^{ij}(u)$,

5) the family of tensors of type (1,1) (i.e., *affinors*) $(w_n)^i_j(u)$, $1 \leq n \leq L$, is commutative: $[w_n(u), w_m(u)] = 0$.

A Hamiltonian operator of the form (1.1) exactly corresponds to an N-dimensional surface with flat normal bundle embedded in a pseudo-Euclidean space E^{N+L}. Here,

the covariant metric $g_{ij}(u)$, for which $g_{is}(u)g^{sj}(u) = \delta_i^j$, is the first fundamental form, and the affinors $w_n(u)$, $1 \leq n \leq L$, are the corresponding Weingarten operators of this embedded surface ($g_{is}(u)(w_n)^s_j(u)$ are the corresponding second fundamental forms). Correspondingly, the relations 2)–4) are the Gauss–Peterson–Codazzi equations for an N-dimensional surface with flat normal bundle embedded in a pseudo-Euclidean space E^{N+L} [1]. The relations 5) are equivalent to the Ricci equations for this embedded surface.

Taking into account the further applications to arbitrary Frobenius manifolds, we prefer to consider the general nonlocal Hamiltonian operators of hydrodynamic type in the form

$$P^{ij} = g^{ij}(u(x))\frac{d}{dx} + b^{ij}_k(u(x))\, u^k_x + \sum_{m=1}^{L}\sum_{n=1}^{L} \mu^{mn}(w_m)^i_k(u(x))u^k_x \left(\frac{d}{dx}\right)^{-1} \circ (w_n)^j_s(u(x))u^s_x, \tag{1.3}$$

where $\det(g^{ij}(u)) \neq 0$, μ^{mn} is an arbitrary nondegenerate symmetric constant matrix. It is obvious that considering linear transformations in the vector space of the affinors $w_n(u)$, $1 \leq n \leq L$, i.e., changing in (1.3) all the affinors $w_n(u)$ to $c^l_n \widetilde{w}_l(u)$, where c^l_n is an arbitrary nondegenerate constant matrix, $w_n(u) = c^l_n \widetilde{w}_l(u)$, any operator of the form (1.3) can be reduced to the form (1.1) and conversely. Among all the conditions 1)–5) for the Hamiltonian property of the operator (1.1), these transformations change only the condition 4) for the Riemannian curvature tensor of the metric. The condition 4) for the operator (1.3) takes the form

$$R^{ij}_{kl}(u) = \sum_{m=1}^{L}\sum_{n=1}^{L} \mu^{mn}\left((w_m)^i_l(u)(w_n)^j_k(u) - (w_m)^j_l(u)(w_n)^i_k(u)\right),$$

all the other conditions 1)–3) and 5) for the Hamiltonian property do not change.

We write all the relations for the coefficients of the nonlocal Hamiltonian operator (1.3) in a form convenient for further use.

Lemma 1.1 *The operator (1.3) is Hamiltonian if and only if its coefficients satisfy the relations*

$$g^{ij} = g^{ji}, \tag{1.4}$$

$$\frac{\partial g^{ij}}{\partial u^k} = b^{ij}_k + b^{ji}_k, \tag{1.5}$$

$$g^{is}b^{jk}_s = g^{js}b^{ik}_s, \tag{1.6}$$

$$g^{is}(w_n)^j_s = g^{js}(w_n)^i_s, \tag{1.7}$$

$$(w_n)^i_s (w_m)^s_j = (w_m)^i_s (w_n)^s_j, \tag{1.8}$$

$$g^{is} g^{jr} \frac{\partial (w_n)^k_r}{\partial u^s} - g^{jr} b^{ik}_s (w_n)^s_r = g^{js} g^{ir} \frac{\partial (w_n)^k_r}{\partial u^s} - g^{ir} b^{jk}_s (w_n)^s_r, \tag{1.9}$$

$$g^{is} \left(\frac{\partial b^{jk}_s}{\partial u^r} - \frac{\partial b^{jk}_r}{\partial u^s} \right) + b^{ij}_s b^{sk}_r - b^{ik}_s b^{sj}_r = \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} g^{is} \left((w_m)^j_r (w_n)^k_s - (w_m)^j_s (w_n)^k_r \right). \tag{1.10}$$

2 Pencil of Hamiltonian operators

Let us consider the important special case of the nonlocal Hamiltonian operators of the form (1.3) when the metric $g^{ij}(u)$ is flat. We recall that any flat metric uniquely defines a local Hamiltonian operator of hydrodynamic type (i.e., a Hamiltonian operator of the form (1.3) with zero affinors) or, in other words, a Dubrovin–Novikov Hamiltonian operator [2]. In this paper, we prove that for any flat metric there is also a remarkable class of nonlocal Hamiltonian operators of hydrodynamic type with this flat metric and nontrivial affinors, and moreover, these Hamiltonian operators have important applications in the theory of Frobenius manifolds and integrable hierarchies.

First of all, we note the following important property of nonlocal Hamiltonian operators of hydrodynamic type with flat metrics. We recall that two Hamiltonian operators are said to be *compatible* if any linear combination of these Hamiltonian operators is also Hamiltonian [4], i.e., they form a *pencil of Hamiltonian operators*.

Lemma 2.1 *The metric $g^{ij}(u)$ of a Hamiltonian operator of the form (1.3) is flat if and only if this operator defines in fact a pencil of compatible Hamiltonian operators*

$$\begin{aligned} P^{ij}_{\lambda_1,\lambda_2} &= \lambda_1 \left(g^{ij}(u(x)) \frac{d}{dx} + b^{ij}_k (u(x))\, u^k_x \right) + \\ &+ \lambda_2 \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} (w_m)^i_k (u(x)) u^k_x \left(\frac{d}{dx} \right)^{-1} \circ (w_n)^j_s (u(x)) u^s_x, \end{aligned} \tag{2.1}$$

where λ_1 and λ_2 are arbitrary constants.

Proof. Actually, if the operator (1.3) is Hamiltonian, then its coefficients satisfy the relations (1.4)–(1.10). It is obvious that in this case the relations (1.4)–(1.9) for the operator (2.1) are always satisfied for any constants λ_1 and λ_2, and the relation

(1.10) is satisfied for any constants λ_1 and λ_2 if and only if the left and the right parts of this relation are equal to zero identically.

It follows from the relations (1.4)–(1.6) for the Hamiltonian operator (1.3) that the Riemannian curvature tensor of the metric $g^{ij}(u)$ has the form

$$R^{ijk}_r(u) = g^{is}(u)R^{jk}_{sr}(u) = g^{is}(u)\left(\frac{\partial b^{jk}_s}{\partial u^r} - \frac{\partial b^{jk}_r}{\partial u^s}\right) + b^{ij}_s(u)b^{sk}_r(u) - b^{ik}_s(u)b^{sj}_r(u). \quad (2.2)$$

Consequently, if the metric $g^{ij}(u)$ of a Hamiltonian operator of the form (1.3) is flat, i.e., $R^{ijk}_r(u) = 0$, then the relation (1.10) becomes

$$\sum_{m=1}^{L}\sum_{n=1}^{L}\mu^{mn}g^{is}\left((w_m)^j_r(u)(w_n)^k_s(u) - (w_m)^j_s(u)(w_n)^k_r(u)\right) = 0.$$

Thus, the metric $g^{ij}(u)$ of a Hamiltonian operator of the form (1.3) is flat if and only if the left and the right parts of the relation (1.10) for the Hamiltonian operator (1.3) are equal to zero identically, and in this case the left and the right parts of the relation (1.10) for the operator (2.1) are also equal to zero identically for any constants λ_1 and λ_2, i.e., we get a pencil of compatible Hamiltonian operators (2.1). Note also that for any pencil of Hamiltonian operators $P^{ij}_{\lambda_1,\lambda_2}$ (2.1) it follows immediately from the Dubrovin–Novikov theorem [2] for the local operator that the metric $g^{ij}(u)$ is flat. Lemma 2.1 is proved.

Thus, if the metric $g^{ij}(u)$ of a Hamiltonian operator of the form (1.3) is flat, then the operator

$$P^{ij}_{0,1} = \sum_{m=1}^{L}\sum_{n=1}^{L}\mu^{mn}(w_m)^i_k(u(x))u^k_x\left(\frac{d}{dx}\right)^{-1}\circ(w_n)^j_s(u(x))u^s_x \quad (2.3)$$

is also a Hamiltonian operator, and moreover, in this case, this Hamiltonian operator is always compatible with the local Hamiltonian operator of hydrodynamic type (the Dubrovin–Novikov operator)

$$P^{ij}_{1,0} = g^{ij}(u(x))\frac{d}{dx} + b^{ij}_k(u(x))\,u^k_x. \quad (2.4)$$

The compatible Hamiltonian operators (2.3) and (2.4) always generate the corresponding integrable bi-Hamiltonian hierarchies. We construct these integrable hierarchies further in this paper.

3 Integrability of structural flows

We recall that the systems of hydrodynamic type

$$u^i_{t_n} = (w_n)^i_j(u)u^j_x, \quad 1 \leq n \leq L, \tag{3.1}$$

are called *structural flows* of the nonlocal Hamiltonian operator of hydrodynamic type (1.3) (see [1], [5]).

Lemma 3.1 *For any nonlocal Hamiltonian operator of hydrodynamic type with a flat metric all the structural flows (3.1) are always commuting integrable bi-Hamiltonian systems of hydrodynamic type.*

Proof. As was proved by Maltsev and Novikov in [5] (see also [1]), the structural flows of any nonlocal Hamiltonian operator of hydrodynamic type (1.3) are always Hamiltonian with respect to this Hamiltonian operator. Let us consider an arbitrary nonlocal Hamiltonian operator of hydrodynamic type (1.3) with a flat metric $g^{ij}(u)$ and the pencil of compatible Hamiltonian operators (2.1) corresponding to this Hamiltonian operator. The corresponding structural flows must be Hamiltonian with respect to any of the operators of the Hamiltonian pencil (2.1) and, consequently, they are integrable bi-Hamiltonian systems.

4 Description of nonlocal Hamiltonian operators of hydrodynamic type with flat metrics

Let us describe all the nonlocal Hamiltonian operators of hydrodynamic type with flat metrics.

The form of the Hamiltonian operator (1.3) is invariant with respect to local changes of coordinates, and also all the coefficients of the operator are transformed as the corresponding differential-geometric objects. Since the metric is flat, there exist local coordinates in which the metric is reduced to a constant form η^{ij}, $\eta^{ij} = \text{const}$, $\det \eta^{ij} \neq 0$, $\eta^{ij} = \eta^{ji}$. In these local coordinates all the coefficients of the Levi-Civita connection are equal to zero, and the Hamiltonian operator has the form:

$$\widetilde{P}^{ij} = \eta^{ij}\frac{d}{dx} + \sum_{m=1}^{L}\sum_{n=1}^{L} \mu^{mn}(\widetilde{w}_m)^i_k(u(x))u^k_x \left(\frac{d}{dx}\right)^{-1} \circ (\widetilde{w}_n)^j_s(u(x))u^s_x. \tag{4.1}$$

Theorem 4.1 *The operator (4.1), where η^{ij} and μ^{mn} are arbitrary nondegenerate symmetric constant matrices, is Hamiltonian if and only if there exist functions $\psi_n(u)$, $1 \le n \le L$, such that*

$$(\widetilde{w}_n)^i_j(u) = \eta^{is} \frac{\partial^2 \psi_n}{\partial u^s \partial u^j}, \tag{4.2}$$

and also the following relations are fulfilled:

$$\sum_{m=1}^{N} \sum_{n=1}^{N} \eta^{mn} \frac{\partial^2 \psi_j}{\partial u^i \partial u^m} \frac{\partial^2 \psi_k}{\partial u^n \partial u^l} = \sum_{m=1}^{N} \sum_{n=1}^{N} \eta^{mn} \frac{\partial^2 \psi_k}{\partial u^i \partial u^m} \frac{\partial^2 \psi_j}{\partial u^n \partial u^l}, \tag{4.3}$$

$$\sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \frac{\partial^2 \psi_m}{\partial u^i \partial u^j} \frac{\partial^2 \psi_n}{\partial u^k \partial u^l} = \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \frac{\partial^2 \psi_m}{\partial u^i \partial u^k} \frac{\partial^2 \psi_n}{\partial u^j \partial u^l}. \tag{4.4}$$

Proof. The relations (1.4)–(1.6) for any operator of the form (4.1) are automatically fulfilled, and the relation (1.9) for any operator of the form (4.1) has the form

$$\frac{\partial (\widetilde{w}_n)^k_r}{\partial u^s} = \frac{\partial (\widetilde{w}_n)^k_s}{\partial u^r}, \tag{4.5}$$

and, consequently, there locally exist functions $\varphi^i_n(u)$, $1 \le i \le N$, $1 \le n \le L$, such that

$$(\widetilde{w}_n)^i_j(u) = \frac{\partial \varphi^i_n}{\partial u^j}. \tag{4.6}$$

The relation (1.7) becomes

$$\eta^{is} \frac{\partial \varphi^j_n}{\partial u^s} = \eta^{js} \frac{\partial \varphi^i_n}{\partial u^s} \tag{4.7}$$

or, equivalently,

$$\frac{\partial \left(\eta_{is} \varphi^s_n\right)}{\partial u^j} = \frac{\partial \left(\eta_{js} \varphi^s_n\right)}{\partial u^i}, \tag{4.8}$$

where η_{ij} is the matrix that is the inverse of the matrix η^{ij}: $\eta_{is}\eta^{sj} = \delta^j_i$. It follows from the relation (4.8) that there locally exist functions $\psi_n(u)$, $1 \le n \le L$, such that

$$\eta_{is} \varphi^s_n = \frac{\partial \psi_n}{\partial u^i}. \tag{4.9}$$

Thus,

$$\varphi^i_n = \eta^{is} \frac{\partial \psi_n}{\partial u^s}, \quad (\widetilde{w}_n)^i_j(u) = \eta^{is} \frac{\partial^2 \psi_n}{\partial u^s \partial u^j}. \tag{4.10}$$

In this case the relations (1.8) and (1.10) become (4.3) and (4.4) respectively. Theorem is proved.

Corollary 4.1 *If the functions $\psi_n(u)$, $1 \le n \le L$, are a solution of the system of the equations (4.3), (4.4), then the systems of hydrodynamic type*

$$u^i_{t_n} = \eta^{is} \frac{\partial^2 \psi_n}{\partial u^s \partial u^j} u^j_x, \qquad 1 \le n \le L, \tag{4.11}$$

are always commuting integrable bi-Hamiltonian systems of hydrodynamic type. Moreover, in this case the operator

$$M_1^{ij} = \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \eta^{ip} \eta^{jr} \frac{\partial^2 \psi_m}{\partial u^p \partial u^k} u^k_x \left(\frac{d}{dx} \right)^{-1} \circ \frac{\partial^2 \psi_n}{\partial u^r \partial u^s} u^s_x \tag{4.12}$$

is also a Hamiltonian operator, and also this Hamiltonian operator is always compatible with the constant Hamiltonian operator

$$M_2^{ij} = \eta^{ij} \frac{d}{dx}. \tag{4.13}$$

In arbitrary local coordinates we obtain the following description of all nonlocal Hamiltonian operators of hydrodynamic type with flat metrics.

Theorem 4.2 *The operator (1.3) with a flat metric $g^{ij}(u)$ is Hamiltonian if and only if $b^{ij}_k(u) = -g^{is}(u)\Gamma^j_{sk}(u)$, where $\Gamma^j_{sk}(u)$ is the flat connection generated by the flat metric $g^{ij}(u)$, and there locally exist functions $\Phi_n(u)$, $1 \le n \le L$, such that*

$$(w_n)^i_j(u) = \nabla^i \nabla_j \Phi_n, \tag{4.14}$$

and also the following relations are fulfilled:

$$\sum_{n=1}^{N} \nabla^n \nabla_i \Phi_j \nabla_n \nabla_l \Phi_k = \sum_{n=1}^{N} \nabla^n \nabla_i \Phi_k \nabla_n \nabla_l \Phi_j, \tag{4.15}$$

$$\sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \nabla_i \nabla_j \Phi_m \nabla_k \nabla_l \Phi_n = \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \nabla_i \nabla_k \Phi_m \nabla_j \nabla_l \Phi_n, \tag{4.16}$$

where ∇_i is the operator of covariant differentiation defined by the flat connection $\Gamma^i_{jk}(u)$ generated by the metric $g^{ij}(u)$, $\nabla^i = g^{is}(u)\nabla_s$. In particular, in this case the operator

$$\begin{aligned} M^{ij}_{\lambda_1,\lambda_2} \quad &= \quad \lambda_1 \left(g^{ij}(u(x)) \frac{d}{dx} - g^{is}(u(x)) \Gamma^j_{sk}(u(x))\, u^k_x \right) + \\ &+ \quad \lambda_2 \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \nabla^i \nabla_k \Phi_m u^k_x \left(\frac{d}{dx} \right)^{-1} \circ \nabla^j \nabla_s \Phi_n u^s_x \end{aligned} \tag{4.17}$$

is a Hamiltonian operator for any constants λ_1 and λ_2, and the systems of hydrodynamic type

$$u^i_{t_n} = \nabla^i \nabla_j \Phi_n u^j_x, \qquad 1 \le n \le L, \tag{4.18}$$

are always commuting integrable bi-Hamiltonian systems of hydrodynamic type.

5 Integrable hierarchies

Let us consider the recursion operator R^i_j corresponding to the compatible Hamiltonian operators (4.12) and (4.13):

$$R^i_j = \left(M_1 (M_2)^{-1}\right)^i_j = \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \eta^{ip} \frac{\partial^2 \psi_m}{\partial u^p \partial u^k} u^k_x \left(\frac{d}{dx}\right)^{-1} \circ \frac{\partial^2 \psi_n}{\partial u^j \partial u^s} u^s_x \left(\frac{d}{dx}\right)^{-1}. \tag{5.1}$$

Let us apply the constructed recursion operator (5.1) to the system of translations with respect to x, i.e., to the system

$$u^i_t = u^i_x. \tag{5.2}$$

Any system in the hierarchy

$$u^i_{t_s} = (R^s)^i_j u^j_x, \qquad s \in \mathbf{Z}, \tag{5.3}$$

is a multi-Hamiltonian integrable system of hydrodynamic type. In particular, any system of the form

$$u^i_{t_1} = R^i_j u^j_x, \tag{5.4}$$

i.e., the system

$$u^i_{t_1} = \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \eta^{ip} \frac{\partial^2 \psi_m}{\partial u^p \partial u^k} u^k_x \left(\frac{d}{dx}\right)^{-1} \circ \frac{\partial^2 \psi_n}{\partial u^j \partial u^s} u^j u^s_x, \tag{5.5}$$

is integrable.

Since

$$\frac{\partial}{\partial u^r}\left(\frac{\partial^2 \psi_n}{\partial u^j \partial u^s} u^j\right) = \frac{\partial^3 \psi_n}{\partial u^j \partial u^s \partial u^r} u^j + \frac{\partial^2 \psi_n}{\partial u^r \partial u^s} = \frac{\partial}{\partial u^s}\left(\frac{\partial^2 \psi_n}{\partial u^j \partial u^r} u^j\right), \tag{5.6}$$

there locally exist functions $F_n(u)$, $1 \le n \le L$, such that

$$\frac{\partial^2 \psi_n}{\partial u^j \partial u^s} u^j = \frac{\partial F_n}{\partial u^s}, \qquad F_n = \frac{\partial \psi_n}{\partial u^j} u^j - \psi_n. \tag{5.7}$$

Thus, the system of hydrodynamic type (5.5) has a local form:

$$u^i_{t_1} = \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \eta^{ip} F_n(u) \frac{\partial^2 \psi_m}{\partial u^p \partial u^k} u^k_x. \tag{5.8}$$

This system of hydrodynamic type is bi-Hamiltonian with respect to the compatible Hamiltonian operators (4.12) and (4.13):

$$u^i_{t_1} = \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \eta^{ip} \eta^{jr} \frac{\partial^2 \psi_m}{\partial u^p \partial u^k} u^k_x \left(\frac{d}{dx}\right)^{-1} \left(\frac{\partial^2 \psi_n}{\partial u^r \partial u^s} u^s_x \frac{\delta H_1}{\delta u^j(x)}\right), \tag{5.9}$$

$$H_1 = \int h_1(u(x)) dx, \qquad h_1(u(x)) = \frac{1}{2} \eta_{ij} u^i(x) u^j(x), \tag{5.10}$$

$$u^i_{t_1} = \eta^{ij} \frac{d}{dx} \frac{\delta H_2}{\delta u^j(x)}, \qquad H_2 = \int h_2(u(x)) dx, \tag{5.11}$$

since in our case there always exists locally a function $h_2(u)$ such that

$$\sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \frac{\partial^2 \psi_m}{\partial u^j \partial u^k} F_n(u) = \frac{\partial^2 h_2}{\partial u^j \partial u^k}. \tag{5.12}$$

Actually, we have

$$\begin{aligned}
&\frac{\partial}{\partial u^i} \left(\sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \frac{\partial^2 \psi_m}{\partial u^j \partial u^k} F_n(u) \right) = \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \frac{\partial^3 \psi_m}{\partial u^i \partial u^j \partial u^k} F_n(u) + \\
&+ \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \frac{\partial^2 \psi_m}{\partial u^j \partial u^k} \frac{\partial F_n}{\partial u^i} = \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \frac{\partial^3 \psi_m}{\partial u^i \partial u^j \partial u^k} F_n(u) + \\
&+ \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \frac{\partial^2 \psi_m}{\partial u^j \partial u^k} \frac{\partial^2 \psi_n}{\partial u^i \partial u^s} u^s = \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \frac{\partial^3 \psi_m}{\partial u^i \partial u^j \partial u^k} F_n(u) + \\
&+ \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \frac{\partial^2 \psi_m}{\partial u^j \partial u^i} \frac{\partial^2 \psi_n}{\partial u^k \partial u^s} u^s,
\end{aligned} \tag{5.13}$$

where we have used the relation (4.4). Consequently, by virtue of symmetry with respect to the indices i and j we obtain

$$\frac{\partial}{\partial u^i} \left(\sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \frac{\partial^2 \psi_m}{\partial u^j \partial u^k} F_n(u) \right) = \frac{\partial}{\partial u^j} \left(\sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \frac{\partial^2 \psi_m}{\partial u^i \partial u^k} F_n(u) \right), \tag{5.14}$$

i.e., there locally exist functions $a_k(u)$, $1 \le k \le N$, such that

$$\sum_{m=1}^{L}\sum_{n=1}^{L} \mu^{mn} \frac{\partial^2 \psi_m}{\partial u^j \partial u^k} F_n(u) = \frac{\partial a_k}{\partial u^j}. \tag{5.15}$$

By virtue of symmetry with respect to the indices j and k we get

$$\frac{\partial a_k}{\partial u^j} = \frac{\partial a_j}{\partial u^k}, \tag{5.16}$$

i.e., there locally exists a function $h_2(u)$ such that

$$a_k(u) = \frac{\partial h_2}{\partial u^k}. \tag{5.17}$$

Thus,

$$\sum_{m=1}^{L}\sum_{n=1}^{L} \mu^{mn} \frac{\partial^2 \psi_m}{\partial u^j \partial u^k} F_n(u) = \frac{\partial a_k}{\partial u^j} = \frac{\partial^2 h_2}{\partial u^j \partial u^k}. \tag{5.18}$$

Consider the following equation of the integrable hierarchy (5.3).

$$\begin{aligned} u^i_{t_2} &= (R^2)^i_j u^j_x = \\ &= \sum_{m=1}^{L}\sum_{n=1}^{L} \mu^{mn} \eta^{ip} \frac{\partial^2 \psi_m}{\partial u^p \partial u^k} u^k_x \left(\frac{d}{dx}\right)^{-1} \circ \frac{\partial^2 \psi_n}{\partial u^j \partial u^s} u^s_x \left(\frac{d}{dx}\right)^{-1} \circ \eta^{jr} \frac{d}{dx} \frac{\delta H_2}{\delta u^r(x)} = \\ &= \sum_{m=1}^{L}\sum_{n=1}^{L} \mu^{mn} \eta^{ip} \frac{\partial^2 \psi_m}{\partial u^p \partial u^k} u^k_x \left(\frac{d}{dx}\right)^{-1} \circ \frac{\partial^2 \psi_n}{\partial u^j \partial u^s} u^s_x \eta^{jr} \frac{\partial h_2}{\partial u^r}. \end{aligned} \tag{5.19}$$

We prove that in our case there always exist locally functions $G_n(u)$, $1 \le n \le L$, such that

$$\frac{\partial^2 \psi_n}{\partial u^j \partial u^s} \eta^{jr} \frac{\partial h_2}{\partial u^r} = \frac{\partial G_n}{\partial u^s}. \tag{5.20}$$

Actually, we have

$$\begin{aligned} \frac{\partial}{\partial u^p} \left(\frac{\partial^2 \psi_n}{\partial u^j \partial u^s} \eta^{jr} \frac{\partial h_2}{\partial u^r} \right) &= \frac{\partial^3 \psi_n}{\partial u^j \partial u^s \partial u^p} \eta^{jr} \frac{\partial h_2}{\partial u^r} + \frac{\partial^2 \psi_n}{\partial u^j \partial u^s} \eta^{jr} \frac{\partial^2 h_2}{\partial u^r \partial u^p} = \\ &= \frac{\partial^3 \psi_n}{\partial u^j \partial u^s \partial u^p} \eta^{jr} \frac{\partial h_2}{\partial u^r} + \frac{\partial^2 \psi_n}{\partial u^j \partial u^s} \eta^{jr} \left(\sum_{k=1}^{L}\sum_{l=1}^{L} \mu^{kl} \frac{\partial^2 \psi_k}{\partial u^r \partial u^p} F_l(u) \right) = \\ &= \frac{\partial^3 \psi_n}{\partial u^j \partial u^s \partial u^p} \eta^{jr} \frac{\partial h_2}{\partial u^r} + \sum_{k=1}^{L}\sum_{l=1}^{L} \mu^{kl} \eta^{jr} \frac{\partial^2 \psi_n}{\partial u^s \partial u^j} \frac{\partial^2 \psi_k}{\partial u^r \partial u^p} F_l(u) = \end{aligned}$$

$$= \frac{\partial^3 \psi_n}{\partial u^j \partial u^s \partial u^p} \eta^{jr} \frac{\partial h_2}{\partial u^r} + \sum_{k=1}^{L} \sum_{l=1}^{L} \mu^{kl} \eta^{jr} \frac{\partial^2 \psi_k}{\partial u^s \partial u^j} \frac{\partial^2 \psi_n}{\partial u^r \partial u^p} F_l(u) =$$

$$= \frac{\partial^3 \psi_n}{\partial u^j \partial u^s \partial u^p} \eta^{jr} \frac{\partial h_2}{\partial u^r} + \sum_{k=1}^{L} \sum_{l=1}^{L} \mu^{kl} \eta^{jr} \frac{\partial^2 \psi_k}{\partial u^s \partial u^r} \frac{\partial^2 \psi_n}{\partial u^j \partial u^p} F_l(u), \tag{5.21}$$

where we have used the relation (4.3) and the symmetry of the matrix η^{jr}. Therefore, it is proved that the expression under consideration is symmetric with respect to the indices p and s, i.e.,

$$\frac{\partial}{\partial u^p} \left(\frac{\partial^2 \psi_n}{\partial u^j \partial u^s} \eta^{jr} \frac{\partial h_2}{\partial u^r} \right) = \frac{\partial}{\partial u^s} \left(\frac{\partial^2 \psi_n}{\partial u^j \partial u^p} \eta^{jr} \frac{\partial h_2}{\partial u^r} \right). \tag{5.22}$$

Consequently, there locally exist functions $G_n(u)$, $1 \leq n \leq L$, such that the relation (5.20) is fulfilled, and therefore, it is proved that the second flow of the integrable hierarchy (5.3) has the form of a local system of hydrodynamic type

$$u^i_{t_2} = \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \eta^{ip} G_n(u) \frac{\partial^2 \psi_m}{\partial u^p \partial u^k} u^k_x. \tag{5.23}$$

Repeating the foregoing arguments exactly, we prove by induction that if the functions $\psi_n(u)$, $1 \leq n \leq L$, are a solution of the system of equations (4.3), (4.4), then for any $s \geq 1$ and for the corresponding function $h_s(u(x))$ (starting from the function $h_1(u(x)) = \frac{1}{2} \eta_{ij} u^i(x) u^j(x)$) there always exist locally functions $F_n^{(s)}(u)$, $1 \leq n \leq L$, such that

$$\frac{\partial^2 \psi_n}{\partial u^j \partial u^p} \eta^{jr} \frac{\partial h_s}{\partial u^r} = \frac{\partial F_n^{(s)}}{\partial u^p}, \tag{5.24}$$

and there always exists locally a function $h_{s+1}(u(x))$ such that

$$\sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \frac{\partial^2 \psi_m}{\partial u^j \partial u^k} F_n^{(s)}(u) = \frac{\partial^2 h_{s+1}}{\partial u^j \partial u^k}. \tag{5.25}$$

Above we have proved that this statement is true for $s = 1$ (in this case, in particular, $F_n^{(1)} = F_n$, $F_n^{(2)} = G_n$). In just the same way it is proved that if this statement is true for $s = K \geq 1$, then it is true also for $s = K + 1$ (see (5.13)–(5.18) and (5.21), (5.22)). Thus, it is proved that for any $s \geq 1$ the corresponding flow of the integrable hierarchy (5.3) has the form of a local system of hydrodynamic type

$$u^i_{t_s} = \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \eta^{ip} F_n^{(s)}(u) \frac{\partial^2 \psi_m}{\partial u^p \partial u^k} u^k_x. \tag{5.26}$$

All the flows of the hierarchy (5.3) are commuting integrable bi-Hamiltonian systems of hydrodynamic type with an infinite family of local integrals in involution with respect to both the Poisson brackets:

$$u^i_{t_s} = M_1^{ij} \frac{\delta H_s}{\delta u^j(x)} = \{u^i, H_s\}_1, \qquad H_s = \int h_s(u(x))dx, \tag{5.27}$$

$$u^i_{t_s} = M_2^{ij} \frac{\delta H_{s+1}}{\delta u^j(x)} = \{u^i, H_{s+1}\}_2, \qquad H_{s+1} = \int h_{s+1}(u(x))dx, \tag{5.28}$$

$$\{H_p, H_r\}_1 = 0, \qquad \{H_p, H_r\}_2 = 0, \tag{5.29}$$

and the densities of the Hamiltonians $h_s(u(x))$ are related by the reccurences (5.24), (5.25), which are always resolvable in our case.

6 Locality and integrability of Hamiltonian systems with nonlocal Poisson brackets

Let the functions $\psi_n(u)$, $1 \le n \le L$, be a solution of the system of equations (4.3), (4.4), i.e., in particular, the nonlocal operator M_1^{ij} (4.12) is Hamiltonian. Let us consider the Hamiltonian system

$$u^i_t = M_1^{ij} \frac{\delta H}{\delta u^j(x)} \tag{6.1}$$

with an arbitrary Hamiltonian of hydrodynamic type

$$H = \int h(u(x))dx. \tag{6.2}$$

As was proved by Ferapontov in [1], a Hamiltonian system with a nonlocal Hamiltonian operator of hydrodynamic type (1.1) and a Hamiltonian of hydrodynamic type is local if and only if the Hamiltonian is an integral of all the structural flows of the Hamiltonian operator. This statement is also true for Hamiltonian operators of the form (2.3), and, moreover, generally for any weakly nonlocal Hamiltonian operators (see [5]). We prove that for the nonlocal Hamiltonian operators M_1^{ij} (4.12) constructed in this paper this condition for Hamiltonians is sufficient for integrability, i.e., all local Hamiltonian systems (6.1), (6.2) are integrable bi-Hamiltonian systems.

Lemma 6.1 *The system (6.1), (6.2) is local if and only if the density $h(u(x))$ of the Hamiltonian satisfies the equations*

$$\frac{\partial^2 \psi_n}{\partial u^j \partial u^s} \eta^{jr} \frac{\partial^2 h}{\partial u^r \partial u^p} = \frac{\partial^2 \psi_n}{\partial u^j \partial u^p} \eta^{jr} \frac{\partial^2 h}{\partial u^r \partial u^s}. \tag{6.3}$$

Proof. Consider a Hamiltonian system (6.1), (6.2):

$$u^i_t = M^{ij}_1 \frac{\delta H}{\delta u^j(x)} = \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \eta^{ip} \frac{\partial^2 \psi_m}{\partial u^p \partial u^k} u^k_x \left(\frac{d}{dx}\right)^{-1} \left(\eta^{jr} \frac{\partial^2 \psi_n}{\partial u^r \partial u^s} u^s_x \frac{\partial h}{\partial u^j}\right). \tag{6.4}$$

The system (6.4) is local if and only if there locally exist functions $P_n(u)$, $1 \leq n \leq L$, such that

$$\frac{\partial^2 \psi_n}{\partial u^j \partial u^s} \eta^{jr} \frac{\partial h}{\partial u^r} = \frac{\partial P_n}{\partial u^s}, \tag{6.5}$$

i.e., if and only if the consistency relation

$$\frac{\partial}{\partial u^p} \left(\frac{\partial^2 \psi_n}{\partial u^j \partial u^s} \eta^{jr} \frac{\partial h}{\partial u^r}\right) = \frac{\partial}{\partial u^s} \left(\frac{\partial^2 \psi_n}{\partial u^j \partial u^p} \eta^{jr} \frac{\partial h}{\partial u^r}\right) \tag{6.6}$$

is fulfilled. Then the system (6.4) takes a local form

$$u^i_t = M^{ij}_1 \frac{\delta H}{\delta u^j(x)} = \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \eta^{ip} \frac{\partial^2 \psi_m}{\partial u^p \partial u^k} P_n(u) u^k_x. \tag{6.7}$$

The consistency relation (6.6) is equivalent to the equations (6.3).

Theorem 6.1 *If a Hamiltonian system (6.1), (6.2) is local, i.e., the density $h(u(x))$ of the Hamiltonian satisfies the equations (6.3), then this system is integrable.*

Proof. In this case the system (6.1), (6.2) takes the form (6.7), (6.5). Let us prove that there always exists locally a function $f(u)$ such that

$$\sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \frac{\partial^2 \psi_m}{\partial u^j \partial u^k} P_n(u) = \frac{\partial^2 f}{\partial u^j \partial u^k}. \tag{6.8}$$

Actually, we have

$$\frac{\partial}{\partial u^i} \left(\sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \frac{\partial^2 \psi_m}{\partial u^j \partial u^k} P_n(u)\right) = \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \frac{\partial^3 \psi_m}{\partial u^i \partial u^j \partial u^k} P_n(u) +$$

$$
+\sum_{m=1}^{L}\sum_{n=1}^{L}\mu^{mn}\frac{\partial^2\psi_m}{\partial u^j\partial u^k}\frac{\partial P_n}{\partial u^i}=\sum_{m=1}^{L}\sum_{n=1}^{L}\mu^{mn}\frac{\partial^3\psi_m}{\partial u^i\partial u^j\partial u^k}P_n(u)+
$$

$$
+\sum_{m=1}^{L}\sum_{n=1}^{L}\mu^{mn}\frac{\partial^2\psi_m}{\partial u^j\partial u^k}\frac{\partial^2\psi_n}{\partial u^i\partial u^p}\eta^{pr}\frac{\partial h}{\partial u^r}=\sum_{m=1}^{L}\sum_{n=1}^{L}\mu^{mn}\frac{\partial^3\psi_m}{\partial u^i\partial u^j\partial u^k}P_n(u)+
$$

$$
+\sum_{m=1}^{L}\sum_{n=1}^{L}\mu^{mn}\frac{\partial^2\psi_m}{\partial u^j\partial u^i}\frac{\partial^2\psi_n}{\partial u^k\partial u^p}\eta^{pr}\frac{\partial h}{\partial u^r}, \tag{6.9}
$$

where we have used the relation (4.4). Consequently, by virtue of symmetry with respect to the indices i and j we get

$$
\frac{\partial}{\partial u^i}\left(\sum_{m=1}^{L}\sum_{n=1}^{L}\mu^{mn}\frac{\partial^2\psi_m}{\partial u^j\partial u^k}P_n(u)\right)=\frac{\partial}{\partial u^j}\left(\sum_{m=1}^{L}\sum_{n=1}^{L}\mu^{mn}\frac{\partial^2\psi_m}{\partial u^i\partial u^k}P_n(u)\right), \tag{6.10}
$$

i.e., there locally exist functions $b_k(u)$, $1\le k\le N$, such that

$$
\sum_{m=1}^{L}\sum_{n=1}^{L}\mu^{mn}\frac{\partial^2\psi_m}{\partial u^j\partial u^k}P_n(u)=\frac{\partial b_k}{\partial u^j}. \tag{6.11}
$$

By virtue of symmetry with respect to the indices j and k we get

$$
\frac{\partial b_k}{\partial u^j}=\frac{\partial b_j}{\partial u^k}, \tag{6.12}
$$

i.e., there locally exists a function $f(u)$ such that

$$
b_k(u)=\frac{\partial f}{\partial u^k}. \tag{6.13}
$$

Thus,

$$
\sum_{m=1}^{L}\sum_{n=1}^{L}\mu^{mn}\frac{\partial^2\psi_m}{\partial u^j\partial u^k}P_n(u)=\frac{\partial b_k}{\partial u^j}=\frac{\partial^2 f}{\partial u^j\partial u^k}. \tag{6.14}
$$

Consequently, the system (6.1), (6.2) in the case under consideration can be presented in the form

$$
u^i_t=\eta^{ij}\frac{\partial^2 f}{\partial u^j\partial u^k}u^k_x=M_2^{ij}\frac{\delta F}{\delta u^j(x)},\qquad F=\int f(u)dx, \tag{6.15}
$$

i.e., it is an integrable bi-Hamiltonian system with the compatible Hamiltonian operators M_1^{ij} (4.12) and M_2^{ij} (4.13).

7 Systems of integrals in involution

The nonlinear equations of the form (4.3) and (6.3) are of independent interest, they have an important significance and very natural interpretation.

Lemma 7.1 *The nonlinear equations (4.3) are equivalent to the condition that the integrals*

$$\Psi_n = \int \psi_n(u(x))dx, \quad 1 \le n \le L, \tag{7.1}$$

are in involution with respect to the Poisson bracket given by the constant Hamiltonian operator M_2^{ij} (4.13):

$$\{\Psi_n, \Psi_m\} = 0. \tag{7.2}$$

Proof. Actually, we have

$$\{\Psi_n, \Psi_m\} = \int \frac{\partial \psi_n}{\partial u^i} \eta^{ij} \frac{d}{dx} \frac{\partial \psi_m}{\partial u^j} dx = \int \frac{\partial \psi_n}{\partial u^i} \eta^{ij} \frac{\partial^2 \psi_m}{\partial u^j \partial u^k} u_x^k dx. \tag{7.3}$$

Consequently, the integrals are in involution, i.e.,

$$\{\Psi_n, \Psi_m\} = 0, \tag{7.4}$$

if and only if there exists a function $S_{nm}(u)$ such that

$$\frac{\partial \psi_n}{\partial u^i} \eta^{ij} \frac{\partial^2 \psi_m}{\partial u^j \partial u^k} = \frac{\partial S_{nm}}{\partial u^k}, \tag{7.5}$$

i.e., if and only if the consistency relation

$$\frac{\partial}{\partial u^l} \left(\frac{\partial \psi_n}{\partial u^i} \eta^{ij} \frac{\partial^2 \psi_m}{\partial u^j \partial u^k} \right) = \frac{\partial}{\partial u^k} \left(\frac{\partial \psi_n}{\partial u^i} \eta^{ij} \frac{\partial^2 \psi_m}{\partial u^j \partial u^l} \right) \tag{7.6}$$

is fulfilled. The consistency relation (7.6) is equivalent to the equations (4.3).

It is proved similarly that the equations (6.3) are equivalent to the condition

$$\{\Psi_n, H\} = 0, \qquad H = \int h(u(x))dx. \tag{7.7}$$

We note that the equations (4.15) are equivalent to the condition that L integrals are in involution with respect to an arbitrary Dubrovin–Novikov bracket (a nondegenerate local Poisson bracket of hydrodynamic type).

Corollary 7.1 *The hamiltonian system (6.1), (6.2) is local if and only if it is generated by a family of $L+1$ integrals in involution with respect to the Poisson bracket given by the constant Hamiltonian operator M_2^{ij} (4.13):*

$$\Psi_n = \int \psi_n(u(x))dx, \quad 1 \leq n \leq L, \quad H = \int h(u(x))dx, \quad \{\Psi_n, \Psi_m\} = 0, \quad \{\Psi_n, H\} = 0. \tag{7.8}$$

In this case the system (6.1), (6.2) is an integrable bi-Hamiltonian system.

The important special class of integrals in involution is generated by the equations of associativity of two-dimensional topological quantum field theory.

Lemma 7.2 *A function $\Phi(u^1, ..., u^N)$ generates a family of N integrals in involution with respect to the Poisson bracket given by the constant Hamiltonian operator M_2^{ij} (4.13):*

$$I_n = \int \frac{\partial \Phi}{\partial u^n}(u(x))dx, \quad \{I_n, I_m\} = 0, \quad 1 \leq n, m \leq N, \tag{7.9}$$

if and only if the function $\Phi(u)$ is a solution of the equations of associativity in two-dimensional topological quantum field theory

$$\sum_{m=1}^{N} \sum_{n=1}^{N} \eta^{mn} \frac{\partial^3 \Phi}{\partial u^i \partial u^j \partial u^m} \frac{\partial^3 \Phi}{\partial u^n \partial u^k \partial u^l} = \sum_{m=1}^{N} \sum_{n=1}^{N} \eta^{mn} \frac{\partial^3 \Phi}{\partial u^i \partial u^k \partial u^m} \frac{\partial^3 \Phi}{\partial u^n \partial u^j \partial u^l}. \tag{7.10}$$

8 Flat submanifolds with flat normal bundle, Hessians and nonlocal Poisson brackets

The nonlinear equations (4.3) and (4.4) describing all nonlocal Hamiltonian operators of hydrodynamic type with flat metrics are exactly equivalent to the conditions that a flat N-dimensional submanifold with flat normal bundle, with the first fundamental form $\eta_{ij} du^i du^j$ and the second fundamental forms $\omega_n(u)$ given by Hessians of L functions $\psi_n(u)$, $1 \leq n \leq L$:

$$\omega_n(u) = \frac{\partial^2 \psi_n}{\partial u^i \partial u^j} du^i du^j,$$

is embedded in an $(N+L)$-dimensional pseudo-Euclidean space. In particular, the equations (4.4) are the Gauss equations for this case, and the equations (4.3) are the Ricci equations (the Peterson–Codazzi–Mainardi equations are fulfilled automatically

in this case). By the Peterson–Bonnet theorem, any solution $\psi_n(u)$, $1 \leq n \leq L$, of the nonlinear system of equations (4.3) and (4.4) defines a unique up to motions N-dimensional submanifold with flat normal bundle, with the first fundamental form $\eta_{ij} du^i du^j$ and the second fundamental forms $\omega_n(u)$ given by Hessians of the functions $\psi_n(u)$, in the $(N+L)$-dimensional pseudo-Euclidean space.

Let us consider an arbitrary N-dimensional flat submanifold with flat normal bundle in an $(N+L)$-dimensional pseudo-Euclidean space. Let $g_{ij}(u) du^i du^j$ be the first fundamental form of this flat submanifold. Then the following corollary of the theory of submanifolds and the Bonnet theorem is true.

Lemma 8.1 *There locally exist functions $\Phi_n(u)$, $1 \leq n \leq L$, such that the second fundamental forms of the submanifold under consideration have the form*

$$\Omega_n(u) = \nabla_i \nabla_j \Phi_n du^i du^j, \tag{8.1}$$

where ∇_i is the operator of covariant differentiation defined by the Levi-Civita connection of the metric $g_{ij}(u)$. In this case the functions $\Phi_n(u)$, $1 \leq n \leq L$, satisfy the Ricci equations (4.15) and the Gauss equations (4.16) for submanifolds (the Peterson–Codazzi–Mainardi equations are fulfilled automatically in this case). Any solution of the system of equations (4.15) and (4.16) defines a unique up to motions N-dimensional submanifold with flat normal bundle, with the first fundamental form $g_{ij}(u) du^i du^j$ and the second fundamental forms $\Omega_n(u)$ (8.1), in the $(N+L)$-dimensional pseudo-Euclidean space.

9 Equations of associativity and nonlocal Poisson brackets

Here, we show that the class of nonlocal Hamiltonian operators of hydrodynamic type with flat metrics is nontrivial and describe a rich and important family of operators of this class. This family is generated by the equations of associativity of two-dimensional topological quantum field theory.

Although the relations (4.3) and (4.4) differ essentially, they are very similar, and the case of the natural reduction when the relations (4.3) and (4.4) simply coincide is of particular interest. Such reduction immediately leads to the equations of associativity of two-dimensional topological field theory.

In fact, the relations (4.3) and (4.4) coincide if $L = N$, $\mu^{mn} = \eta^{mn}$ (or, for example, $\mu^{mn} = c\eta^{mn}$, where c is an arbitrary nonzero constant) and there exists a

function $\Phi(u)$ such that $\psi_n = \partial \Phi / \partial u^n$ for all n. In this case both the relations (4.3) and (4.4) coincide with the equations of associativity of two-dimensional topological field theory for the potential $\Phi(u)$ (7.10).

Thus, any solution $\Phi(u)$ of the equations of associativity (7.10), which are, as is well known, consistent, integrable by the method of the inverse scattering problem and possess a rich set of nontrivial solutions (see [6]), defines a nonlocal Hamiltonian operator of hydrodynamic type with a flat metric:

$$L^{ij} = \eta^{ij} \frac{d}{dx} + \sum_{m=1}^{N} \sum_{n=1}^{N} \eta^{mn} \eta^{ip} \eta^{jr} \frac{\partial^3 \Phi}{\partial u^p \partial u^m \partial u^k} u^k_x \left(\frac{d}{dx} \right)^{-1} \circ \frac{\partial^3 \Phi}{\partial u^r \partial u^n \partial u^s} u^s_x, \tag{9.1}$$

and, moreover, defines a pencil of compatible Hamiltonian operators:

$$L^{ij}_{\lambda_1, \lambda_2} = \lambda_1 \eta^{ij} \frac{d}{dx} + \lambda_2 \sum_{m=1}^{N} \sum_{n=1}^{N} \eta^{mn} \eta^{ip} \eta^{jr} \frac{\partial^3 \Phi}{\partial u^p \partial u^m \partial u^k} u^k_x \left(\frac{d}{dx} \right)^{-1} \circ \frac{\partial^3 \Phi}{\partial u^r \partial u^n \partial u^s} u^s_x, \tag{9.2}$$

where λ_1 and λ_2 are arbitrary constants. In particular, if the function $\Phi(u)$ is an arbitrary solution of the equations of associativity (7.10), then the operator

$$L^{ij}_{0,1} = \sum_{m=1}^{N} \sum_{n=1}^{N} \eta^{mn} \eta^{ip} \eta^{jr} \frac{\partial^3 \Phi}{\partial u^p \partial u^m \partial u^k} u^k_x \left(\frac{d}{dx} \right)^{-1} \circ \frac{\partial^3 \Phi}{\partial u^r \partial u^n \partial u^s} u^s_x \tag{9.3}$$

is a Hamiltonian operator compatible with the constant Hamiltonian operator

$$L^{ij}_{1,0} = \eta^{ij} \frac{d}{dx}. \tag{9.4}$$

Therefore, for any solution of the equations of associativity (7.10) we obtain the corresponding integrable hierarchies (see Section 5).

The metric η^{ij} always defines a nondegenerate invariant symmetric bilinear form in the corresponding Frobenius algebras, namely, commutative associative algebras $A(u)$ in an N-dimensional vector space with a basis $e_1, \ldots, e_N$ and the multiplication (see [6])

$$e_i * e_j = \eta^{ks} \frac{\partial^3 \Phi}{\partial u^s \partial u^i \partial u^j} e_k, \tag{9.5}$$

$$< e_i, e_j >= \eta_{ij}, \qquad < e_i * e_j, e_k >=< e_i, e_j * e_k > . \tag{9.6}$$

In this case the condition of associativity

$$(e_i * e_j) * e_k = e_i * (e_j * e_k) \tag{9.7}$$

in the algebras $A(u)$ is equivalent to the equations (7.10). We recall (see Dubrovin, [6]) that locally on the tangent space in every point of any Frobenius manifold there is a structure of Frobenius algebra (9.5), (9.6), (9.7) defined by a certain solution of the equations of associativity (7.10) and depending on the point smoothly (besides, it is also required the fulfilment of additional conditions on Frobenius manifolds but we do not consider these conditions here; roughly speaking, it is required the presence of a unit and the quasihomogeneity). Thus, for every Frobenius manifold there are nonlocal Hamiltonian operators of the form (9.1) and pencils (9.2) connected to the manifold.

We have considered the nonlocal Hamiltonian operators of the form (1.3) with flat metrics and came to the equations of associativity defining the affinors of such operators. A statement that is in some sense the converse is also true, namely, if all the affinors w_n of a nonlocal Hamiltonian operator (1.3) with $L = N$ are defined by an arbitrary solution $\Phi(u)$ of the equations of associativity

$$\sum_{m=1}^{N}\sum_{n=1}^{N} \mu^{mn} \frac{\partial^3 \Phi}{\partial u^i \partial u^j \partial u^m} \frac{\partial^3 \Phi}{\partial u^n \partial u^k \partial u^l} = \sum_{m=1}^{N}\sum_{n=1}^{N} \mu^{mn} \frac{\partial^3 \Phi}{\partial u^i \partial u^k \partial u^m} \frac{\partial^3 \Phi}{\partial u^n \partial u^j \partial u^l} \tag{9.8}$$

by the formula

$$(w_n)^i_j(u) = \zeta^{is} \xi^r_j \frac{\partial^3 \Phi}{\partial u^n \partial u^s \partial u^r},$$

where ζ^{is}, ξ^r_j are arbitrary nondegenerate constant matrices, then the metric of this Hamiltonian operator must be flat. But, in general, it is not necessarily that this metric will be constant in the local coordinates under consideration.

The structural flows (see [1], [5]) of the nonlocal Hamiltonian operator (9.1) have the form:

$$u^i_{t_n} = \eta^{is} \frac{\partial^3 \Phi}{\partial u^s \partial u^n \partial u^k} u^k_x. \tag{9.9}$$

These systems are integrable bi-Hamiltonian systems of hydrodynamic type and coincide with the primary part of the Dubrovin hierarchy constructed by any solution of the equations of associativity in [6]. The condition of commutation for the structural flows (9.9) is also equivalent to the equations of associativity (7.10).

From the consideration of the previous Section we have that by the Peterson–Bonnet theorem any solution $\Phi(u)$ of the equations of associativity (7.10) defines a unique up to motions N-dimensional submanifold with flat normal bundle, with the first fundamental form $\eta_{ij} du^i du^j$ and the second fundamental forms

$$\omega_n(u) = \frac{\partial^3 \Phi}{\partial u^n \partial u^i \partial u^j} du^i du^j$$

given by the potential $\Phi(u)$, in a $2N$-dimensional pseudo-Euclidean space. Here the signature of the ambient pseudo-Euclidean space is completely defined by the signature of the metric η_{ij}. In this case the equation of associativity coincides with both the Gauss equation and the Ricci equation of the embedded submanifold. Thus, a special class of flat N-dimensional submanifolds with flat normal bundle in a $2N$-dimensional pseudo-Euclidean space locally corresponds to N-dimensional Frobenius manifolds.

A great number of concrete examples of Frobenius manifolds and solutions of the equations of associativity is given in Dubrovin's paper [6]. Here we do not give a great number of examples and an explicit form of the corresponding Hamiltonian operators, flat submanifolds and integrable hierarchies to save place and consider only one simple example from [6] as an illustration. Let $N = 3$ and the metric η_{ij} be antidiagonal

$$(\eta_{ij}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{9.10}$$

and the function $\Phi(u)$ has the form

$$\Phi(u) = \frac{1}{2}(u^1)^2 u^3 + \frac{1}{2} u^1 (u^2)^2 + f(u^2, u^3).$$

In this case e_1 is the unit in the Frobenius algebra (9.5), (9.6), (9.7), and the equation of associativity (7.10) for the function $\Phi(u)$ is equivalent to a remarkable integrable equation of Dubrovin for the function $f(u^2, u^3)$:

$$\frac{\partial^3 f}{\partial (u^3)^3} = \left(\frac{\partial^3 f}{\partial (u^2)^2 \partial u^3} \right)^2 - \frac{\partial^3 f}{\partial (u^2)^3} \frac{\partial^3 f}{\partial u^2 \partial (u^3)^2}. \tag{9.11}$$

This equation is connected to quantum cohomology of projective plane and classical problems of enumerative geometry (see [7]). In particular, all nontrivial polynomial solutions of the equation (9.11) that satisfy the requirement of the quasihomogeneity and, consequently, locally define a structure of Frobenius manifold are described by Dubrovin in [6]:

$$f = \frac{1}{4}(u^2)^2 (u^3)^2 + \frac{1}{60}(u^3)^5, \quad f = \frac{1}{6}(u^2)^3 u^3 + \frac{1}{6}(u^2)^2 (u^3)^3 + \frac{1}{210}(u^3)^7, \tag{9.12}$$

$$f = \frac{1}{6}(u^2)^3 (u^3)^2 + \frac{1}{20}(u^2)^2 (u^3)^5 + \frac{1}{3960}(u^3)^{11}. \tag{9.13}$$

As is shown by the author in [8] (see also [9]), the equation (9.11) is equivalent to the integrable nondiagonalizable homogeneous system of hydrodynamic type

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}_{u^3} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c & 2b & -a \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}_{u^2}, \tag{9.14}$$

$$a = \frac{\partial^3 f}{\partial (u^2)^3}, \quad b = \frac{\partial^3 f}{\partial (u^2)^2 \partial u^3}, \quad c = \frac{\partial^3 f}{\partial u^2 \partial (u^3)^2}. \tag{9.15}$$

In this case the affinors of the nonlocal Hamiltonian operator (9.1) have the form:

$$(w_1)^i_j(u) = \delta^i_j, \quad (w_2)^i_j(u) = \begin{pmatrix} 0 & b & c \\ 1 & a & b \\ 0 & 1 & 0 \end{pmatrix}, \quad (w_3)^i_j(u) = \begin{pmatrix} 0 & c & b^2 - ac \\ 0 & b & c \\ 1 & 0 & 0 \end{pmatrix}. \tag{9.16}$$

References

[1] E. V. Ferapontov, “Differential geometry of nonlocal Hamiltonian operators of hydrodynamic type,” Funkts. Anal. Prilozhen., **25**, No. 3, 37–49 (1991); English transl. Funct. Anal. Appl., **25**, No. 3, 195–204 (1991).

[2] B. A. Dubrovin and S. P. Novikov, “The Hamiltonian formalism of one-dimensional systems of hydrodynamic type and the Bogolyubov–Whitham averaging method,” Dokl. Akad. Nauk SSSR, **270**, No. 4, 781–785 (1983); English transl. Soviet Math. Dokl., **27**, 665–669 (1983).

[3] O. I. Mokhov and E. V. Ferapontov, “Non-local Hamiltonian operators of hydrodynamic type related to metrics of constant curvature,” Usp. Mat. Nauk, **45**, No. 3, 191–192 (1990); English transl. Russian Math. Surveys, **45**, No. 3, 218–219 (1990).

[4] F. Magri, “A simple model of the integrable Hamiltonian equation,” J. Math. Phys., **19**, No. 5, 1156–1162 (1978).

[5] A. Ya. Maltsev and S. P. Novikov, “On the local systems Hamiltonian in the weakly non-local Poisson brackets,” Physica D, **156**, Nos. 1–2, 53–80 (2001); arXiv: nlin.SI/0006030 (2000).

[6] B. Dubrovin, “Geometry of 2D topological field theories,” In: Lect. Notes in Math., Vol. 1620, 1996, pp. 120–348; arXiv: hep-th/9407018 (1994).

[7] M. Kontsevich and Yu. I. Manin, “Gromov–Witten classes, quantum cohomology, and enumerative geometry,” Comm. Math. Phys., **164**, 525–562 (1994).

[8] O. I. Mokhov, “Symplectic and Poisson geometry on loop spaces of manifolds and nonlinear equations,” In: Topics in topology and mathematical physics (S. P. Novikov, ed.), Amer. Math. Soc., Providence, RI, 1995, pp. 121–151; arXiv: hep-th/9503076 (1995).

[9] O. I. Mokhov, Symplectic and Poisson geometry on loop spaces of smooth manifolds and integrable equations, Moscow–Izhevsk, Institute of Computer Studies, 2004 (In Russian); English version: Reviews in Mathematics and Mathematical Physics, Vol. 11, Part 2, Harwood Academic Publishers, 2001.

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